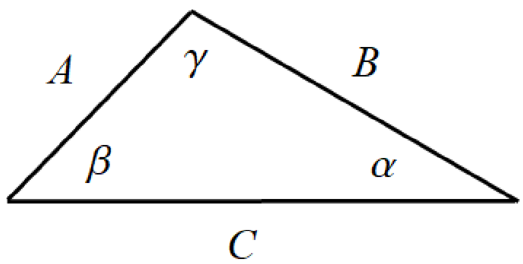
Exercise 33

Prove the statements in Exercises 32 to 34.

> If PQR is a triangle in space and $b > 0$ is a number, then there is a triangle with sides parallel to those of PQR and side lengths b times those of PQR.

Solution

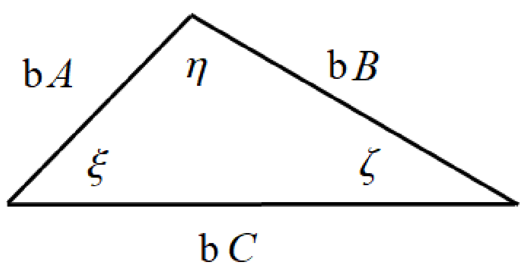
Suppose there's a triangle with known sides and angles.



The law of cosines gives three equations relating the sides with the angles.

$$A^2 = B^2 + C^2 - 2BC\cos\alpha \tag{1}$$
$$B^2 = A^2 + C^2 - 2AC\cos\beta \tag{2}$$
$$C^2 = A^2 + B^2 - 2AB\cos\gamma \tag{3}$$

If the lengths are all multiplied by b, will these angles stay the same?



Use the law of cosines to find out.

$$(bA)^2 = (bB)^2 + (bC)^2 - 2(bB)(bC)\cos\zeta$$
$$(bB)^2 = (bA)^2 + (bC)^2 - 2(bA)(bC)\cos\xi$$
$$(bC)^2 = (bA)^2 + (bB)^2 - 2(bA)(bB)\cos\eta$$

$$b^2A^2 = b^2B^2 + b^2C^2 - 2b^2BC\cos\zeta$$
$$b^2B^2 = b^2A^2 + b^2C^2 - 2b^2AC\cos\xi$$
$$b^2C^2 = b^2A^2 + b^2B^2 - 2b^2AB\cos\eta$$

Divide both sides of each equation by b^2.

$$
\begin{aligned}
A^2 &= B^2 + C^2 - 2BC\cos\zeta \\
B^2 &= A^2 + C^2 - 2AC\cos\xi \\
C^2 &= A^2 + B^2 - 2AB\cos\eta
\end{aligned}
$$

Comparing these to equations (1), (2), and (3) results in

$$
\begin{aligned}
\cos\alpha &= \cos\zeta \\
\cos\beta &= \cos\xi \\
\cos\gamma &= \cos\eta
\end{aligned}
$$

Since all angles are between 0 and 2π, $\alpha = \zeta$, $\beta = \xi$, and $\gamma = \eta$. This means that multiplying a triangle's sides by b results in a new triangle whose sides are parallel with those of the original.